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On the Jacobian Ideal of a Trilinear Form*

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0. INTRODUCTION

Let K be a field and let S be the polynomial ring over K in the three sets of indeterminates $X_0, \dots, X_{n-1}$, $Y_0, \dots, Y_{m-1}$, $Z_0, \dots, Z_{p-1}$. We always assume $n \geq m \geq p \geq 2$. Let

$$A = \sum_{\substack{0 \leq i \leq n-1 \\ 0 \leq j \leq m-1 \\ 0 \leq k \leq p-1}} a_{ijk} X_i Y_j Z_k$$

be a trilinear form in S, and denote by J_A the ideal of S generated by all the partial derivatives of A (the letter J stands for Jacobian). What can be

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said about the ideal J_A? For instance:

QUESTION 1. *What is the height of* J_A?

QUESTION 2. *More precisely, what is the primary decomposition of* J_A?

These questions have attracted the attention of the first author in connection with the notion of hyperdeterminant, the three-dimensional analogue of determinant (as defined by Cayley and rediscovered by Gelfand, Kapranov, and Zelevinsky). For information about hyperdeterminants, we refer the reader to [GKZ]. Here we just recall that, given a three-dimensional matrix (b_{ijk}), $0 \le i \le n-1$, $0 \le j \le m-1$, $0 \le k \le p-1$, $n \ge m \ge p \ge 2$, it makes sense to speak about its hyperdeterminant if, and only if, $n \le m+p-1$. (This is the condition characterizing the "square" three-dimensional matrices.) For a bilinear form

$$C = \sum_{0 \le i, j \le n-1} c_{ij} X_i Y_j \in K[X_0, \dots, X_{n-1} Y_0, \dots, Y_{n-1}],$$

the ideal generated by its partial derivatives has height $2n$ if, and only if, $\det(c_{ij}) \ne 0$. Therefore it is natural to conjecture that, when $n \le m+p-1$, ht $J_A = 2p$ if, and only if, $\operatorname{hyperdet}(a_{ijk}) \ne 0$. (Notice that if we denote by $A_{Z_0}, \dots, A_{Z_{p-1}}$ the partial derivatives of A with respect to the indeterminates $Z_0, \dots, Z_{p-1}$, we have

$$J_A \subseteq \left(Z_0, \dots, Z_{p-1}, A_{Z_0}, \dots, A_{Z_{p-1}}\right),$$

so that ht $J \le 2p$ by Krull's principal ideal theorem.) However, examples show that ht $J_A = 2p$ occurs also in cases in which $\operatorname{hyperdet}(a_{ijk}) = 0$ (see Example 1.16 below). Therefore it becomes interesting to look at the primary decomposition of J_A, in order to have some finer information.

Questions 1 and 2 have proved quite difficult, one reason being that, in general, no formula for the hyperdeterminant is known that would be similar to the usual expansion

$$\det(c_{ij}) = \sum_{\sigma \in S_n} (-1)^{\sigma} c_{0\sigma(0)} \cdots c_{n-1\sigma(n-1)}.$$

Hence it is hard to connect the behavior of J_A to $\operatorname{hyperdet}(a_{ijk})$. But if $n = m+p-1$ (i.e., (a_{ijk}) is of "boundary format"), and $a_{ijk} = 0$ whenever $i \ne j+k$ (i.e., (a_{ijk}) is "diagonal"), then $\operatorname{hyperdet}(a_{ijk})$ is a monomial in the a_{ijk}s such that $i = j+k$ (see Weyman and Zelevinsky [WZ, Section 7]).

Under the assumptions that $n = m+p-1$, and $a_{ijk} \ne 0$ if, and only if, $i = j+k$, we have been able to prove that ht J_A is always $2p$, to determine the minimal primes of J_A and to show that the ideal generated by all indeterminates is associated with J_A. These are the main results of the paper, which will hopefully lead to an understanding of J_A also in cases where A is more general.

Though this work has been motivated by hyperdeterminants, they will not appear in the following. However, as is customary in the theory of hyperdeterminants we start counting indices with 0.

It seem noteworthy that, independently of the link to the notion of hyperdeterminant, the study of J_A is a rather interesting piece of constructive commutative algebra.

1. THE MINIMAL PRIMES OF J_A

We start with the following definition, motivated by the remarks on hyperdeterminants made in the Introduction (nondegenerate means hyperdet $\neq 0$).

DEFINITION 1.1. Let

$$A = \sum_{\substack{0 \le i \le n-1 \\ 0 \le j \le m-1 \\ 0 \le k \le p-1}} a_{ijk} X_i Y_j Z_k$$

be as in the Introduction. We say that A is a *nondegenerate diagonal trilinear form of boundary format*, provided $n = m + p - 1$ and $a_{ijk} \neq 0$ if, and only if, $i = j + k$. (This convention about m, n, and p will be used throughout the paper.)

When A is as in Definition 1.1, the generators of J_A are the following:

$$A_{Z_k} = \sum_{j=0}^{m-1} a_{j+k\,j\,k} X_{j+k} Y_j, \qquad 0 \le k \le p-1,$$

$$A_{Y_j} = \sum_{k=0}^{p-1} a_{j+k\,j\,k} X_{j+k} Z_k, \qquad 0 \le j \le m-1,$$

$$A_{X_i} = \begin{cases} \displaystyle\sum_{k=0}^{i} a_{i\,i-k\,k} Y_{i-k} Z_k, & 0 \le i \le p-1, \\ \displaystyle\sum_{k=0}^{p-1} a_{i\,i-k\,k} Y_{i-k} Z_k, & p-1 \le i \le m-1, \\ \displaystyle\sum_{k=i-m+1}^{p-1} a_{i\,i-k\,k} Y_{i-k} Z_k, & m-1 \le i \le n-1. \end{cases}$$

We shall now describe the minimal primes of the ideal J_A in this case. We shall work out separately the case $p < m$ (Theorem 1.12 below) and the case $p = m$ (Theorem 1.14). In particular we will obtain that if A is a nondegenerate diagonal trilinear form of boundary format, then ht $J_A = 2p$.

In order to proceed, we must recall the notion of symmetric algebra. If M is free of rank g, then the symmetric algebra $S(M)$ is just the polynomial ring in g indeterminates over R: $S(M) \cong R[T_1, \dots, T_g]$. If M has a presentation $F \xrightarrow{(c_{ij})} G \longrightarrow M \longrightarrow 0$ with F and G free of ranks f and g, respectively, then $S(M)$ is isomorphic to $R[T_1, \dots, T_g]/I$, where I is generated by the f elements $\sum_{j=1}^{g} c_{ij}T_j$, $1 \le i \le f$.

In the following $I_t(\phi)$ will denote the ideal generated by the $t \times t$ minors of the matrix ϕ (with the convention that $I_0(\phi)$ is the whole ring).

The following result on symmetric algebras is due to Huneke [H]. More general statements have been found by Avramov [A] and by Simis and Vasconcelos [SV].

PROPOSITION 1.2 (Huneke [H, Theorem 1.1]). *Let R be a Cohen–Macaulay domain and let M be an R-module having a finite free resolution*

$$0 \to R^r \xrightarrow{C} R^s \to M \to 0, \qquad C = (c_{ij}).$$

Then $S(M)$ is a Cohen–Macaulay domain if, and only if, grade$(I_t(C)) \ge r + 2 - t$ for $1 \le t \le r$.

Remark 1.3. If K is a field, $W = K[T_1, \dots, T_r]$ a polynomial ring in r indeterminates, and H the following $p \times (p + r - 1)$ matrix with entries in W,

$$H = \begin{bmatrix} a_{11}T_1 & & & \cdots & a_{1r}T_r & 0 & \cdots & 0 \\ 0 & a_{22}T_1 & & & \cdots & a_{2r+1}T_r & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & & \ddots & 0 \\ 0 & \cdots & 0 & a_{pp}T_1 & & \cdots & & a_{pp+r+1}T_r \end{bmatrix},$$

where the coefficients a_{ij} are nonzero constants, then $I_p(H) = (T_1, \dots, T_r)^p$. (The entries of H that involve the indeterminate T_i appear on the ith diagonal.) This can be proved precisely as in Bruns and Vetter [BV, p. 15] since every a_{ij} is a unit.

We now come to the main results of this section.

PROPOSITION 1.4. *Let $A = \sum a_{ijk} X_i Y_j Z_k$ be a nondegenerate diagonal trilinear form of boundary format. If $Q \supseteq J_A$ is a prime ideal, then Q includes either the ideal $(Z_0, \dots, Z_{p-1}, A_{Z_0}, \dots, A_{Z_{p-1}})$ or the ideal $(Y_0, \dots, Y_{m-1},$*

$A_{Y_0}, \ldots, A_{Y_{m-1}}, I_p(X))$, where $I_p(X)$ denotes the ideal generated by the maximal minors of the $p \times m$ matrix

$$X = \begin{bmatrix} a_{000}X_0 & \cdots & a_{m-1\,m-1\,0}X_{m-1} \\ a_{101}X_1 & \cdots & a_{m\,m-1\,1}X_m \\ \vdots & & \vdots \\ a_{p-1\,0\,p-1}X_{p-1} & \cdots & a_{n-1\,m-1\,p-1}X_{n-1} \end{bmatrix}.$$

(*The entries of X that involve the indeterminate X_i appear on the $(i+1)$st antidiagonal.*)

Proof. Clearly, if $(Z_0, \ldots, Z_{p-1}) \subseteq Q$, we have $(Z_0, \ldots, Z_{p-1}, A_{Z_0}, \ldots, A_{Zp-1}) \subseteq Q$. Suppose now that h, $0 \le h \le p-1$, is the least index with $Z_h \in Q$. For all $i = 0, \ldots, n-1$, call B_{X_i} the part of A_{X_i} formed by all its monomial summands containing one of the indeterminates $Z_0, \ldots, Z_{h-1}$ (set $B_{X_i} = 0$ of $h = 0$). Since $(A_{X_0}, \ldots, A_{X_{n-1}}) \subseteq Q$, the elements

$$\begin{aligned} A_{X_h} - B_{X_h} &= a_{h0h}Y_0Z_h, \\ A_{X_{h+1}} - B_{X_{h+1}} &= a_{h+1\,1\,h}Y_1Z_h + a_{h+1\,0\,h+1}Y_0Z_{h+1}, \\ &\;\;\vdots \\ A_{X_{m-1}} - B_{X_{m-1}} &= a_{m-1\,m-1-h\,h}Y_{m-1-h}Z_h \\ &\qquad + \cdots + a_{m-1\,m-p\,p-1}Y_{m-p}Z_{p-1}, \\ &\;\;\vdots \\ A_{X_{m-1+h}} - B_{X_{m-1+h}} &= a_{m-1+h\,m-1\,h}Y_{m-1}Z_h \\ &\qquad + \cdots + a_{m+1+h\,m-p+h\,p-1}Y_{m-p+h}Z_{p-1} \end{aligned}$$

are all in Q (the number of summands on the right-hand side grows by 1 in each step when the index runs from h to $m-1$ and then it stays constant). It follows easily that Q must contain all the elements $Y_0, \ldots, Y_{m-1}$. Hence $(Y_0, \ldots, Y_{m-1}, A_{Y_0}, \ldots, A_{Y_{p-1}}) \subseteq Q$.

Let Z stand for the $1 \times p$ matrix $[Z_0, \ldots, Z_{p-1}]$, so that $(A_{Y_0}, \ldots, A_{Y_{m-1}}) = I_1(ZX)$. The statement will be proved if we show that $(Z_0, \ldots, Z_{p-1})I_p(X) \subseteq I_1(ZX)$. Hence it is enough to ascertain that, given any $p \times p$ minor f of X, one has $Z_k \cdot f \in I_1(ZX)$ for every $k = 0, \ldots, p-1$. Denote by X' the $p \times p$ submatrix of X such that $f = \det X'$. Then

$$(fZ_0, \ldots, fZ_{p-1}) = I_1(ZX' \operatorname{adj} X') \subseteq I_1(ZX') \subseteq I_1(ZX). \qquad \blacksquare$$

Our first goal is to show that if $p < m$, the minimal primes of J_A are precisely the ideals $(Z_0, \dots, Z_{p-1}, A_{Z_0}, \dots, A_{Z_{p-1}})$ and $(Y_0, \dots, Y_{m-1}, A_{Y_0}, \dots, A_{Y_{m-1}}, I_p(X))$.

PROPOSITION 1.5. *Let $Z = \sum a_{ijk} X_i Y_j Z_k$ be a nondegenerate diagonal trilinear form of boundary format. If $p < m$, the ideal $(Z_0, \dots, Z_{p-1}, A_{Z_0}, \dots, A_{Z_{p-1}})$ is a minimal prime of J_A.*

Proof. J_A is clearly contained in the ideal $(Z_0, \dots, Z_{p-1}, A_{Z_0}, \dots, A_{Z_{p-1}})$. To show that it is a minimal prime of J_A we just need to show that it is prime. To do so it is enough to prove that $(A_{Z_0}, \dots, A_{Z_{p-1}})$ is prime. Let $W = K[Y_0, \dots, Y_{m-1}]$ and let Y denote the following $p \times n$ matrix with entries in W:

$$Y = \begin{bmatrix} a_{000}Y_0 & & \cdots & & a_{m-1\,m-1\,0}Y_{m-1} & 0 & \cdots & 0 \\ 0 & \ddots & & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & & \ddots & 0 \\ 0 & \cdots & 0 & a_{p-1\,0\,p-1}Y_0 & & \cdots & & a_{n-1\,m-1\,p-1}Y_{m-1} \end{bmatrix}.$$

(The entries of Y involving the indeterminate Y_i appear on the $(i+1)$st main diagonal.) Let N be the W-module defined by the short exact sequence

$$0 \to W^p \xrightarrow{Y} W^n \to N \to 0.$$

In view of Remark 1.3, the presentation of the symmetric algebra of N is

$$S(N) \cong \frac{K[X_0, \dots, X_{n-1}, Y_0, \dots, Y_{m-1}]}{\left(A_{Z_0}, \dots, A_{Z_{p-1}}\right)}.$$

By Proposition 1.2, $S(N)$ is a Cohen–Macaulay domain if, and only if, $\operatorname{grade}(I_t(Y)) \geq p + 2 - t$ for $1 \leq t \leq p$. However, by Remark 1.3, $I_p(Y) = (Y_0, \dots, Y_{m-1})^p$, and $I_p(Y) \subseteq I_t(Y)$ for every $t = 1, \dots, p$. Thus $\operatorname{grade}(I_t(Y)) = m \geq p + 1 \geq +2 - t$ for $1 \leq t \leq p$. It follows that $(Z_0, \dots, Z_{p-1}, A_{Z_0}, \dots, A_{Z_{n-1}})$ is a prime ideal of height $2p$. ∎

Our next step is to show that if A is as in Proposition 1.4, then the ideal

$$\left(Y_0, \dots, Y_{m-1}, A_{Y_0}, \dots, A_{Y_{m-1}}, I_p(X)\right)$$

is perfect (not just when $p < m$, but also when $p = m$). Of course, it is enough to show that $(A_{Y_0}, \dots, A_{Y_{m-1}}, I_p(X))$ is perfect. In order to do so, we actually prove something more, namely, that the latter is a geometric m-residual intersection of the ideal $(Z_0, \dots, Z_{p-1})$ with respect to the ideal $(A_{Y_0}, \dots, A_{Y_{m-1}})$.

We recall the definition of geometric m-residual intersection and give some results we shall use.

DEFINITION 1.6. Let R be a Cohen–Macaulay local ring, I an ideal of R, $A = (a_1, \dots, a_m) \subseteq I$, but with $A \neq I$. Set $J = A : I$. If $\operatorname{ht} J \geq m \geq \operatorname{ht} I$, then J is called an *m-residual intersection* of I (with respect to A). If furthermore $I_P = A_P$ for all primes P containing I with $\operatorname{ht} P \leq m$, then we say that J is a *geometric m-residual intersection* of I (with respect to A). We use these notions similarly when (R, M) is a positively graded algebra over a field K with irrelevant maximal ideal M, and I, $a_1, \dots, a_m$, and J are homogeneous.

Let R be a ring and U a matrix with indeterminate entries over R. Then we denote the polynomial ring over R in all the entries simply by $R[U]$, and we use an analogous notation if two or more matrices are involved.

PROPOSITION 1.7 (Bruns *et al.* [BKM, Proposition 4.2]). *Let R be a commutative Noetherian ring, $Z_{1\times p}$ and $U_{p\times m}$ matrices of indeterminates, and $J_{p,m}$ the ideal $I_1(ZU) + I_{\min\{p,m\}}(U) \subset K[Z, U]$. Then*:

(1) *The ideal $J_{p,m}$ has grade at least* $\max\{p, m\}$.

(2) *If $m \geq p$, then $J_{p,m}$ is perfect of grade m.*

(3) *If $m \geq p$, then $J_{p,m} = I_1(ZU)\colon I_1(Z)$.*

(4) *If (R, M) is a Cohen–Macaulay local ring and $m \geq p$, then $J_{p,m}$ is a geometric m-residual intersection of the ideal $I_1(Z)$ in the local ring $R[U, Z]_{(M,U,Z)}$.*

The next proposition will help us in specializing the generic data of Proposition 1.7 to those of our problem.

PROPOSITION 1.8 (Huneke and Ulrich [HU, Proposition 4.2]). *Let R be a Cohen–Macaulay local ring and I an ideal of R. Let m be an integer such that $m \geq \operatorname{ht} I$, $A = (a_1, \dots, a_m) \subseteq I$, $J = A : I$. Let $\underline{l}$ be a regular sequence on both R and R/I and write "$\,'\,$" for reduction modulo $\underline{l}R$. If* $\operatorname{ht}(A' : I') \geq m$, *$R/J$ is Cohen–Macaulay,* $\operatorname{ht} J = m$, *and $(I')_{P'} = (A')_{P'}$ for every prime ideal P' containing I' with* $\operatorname{ht} P' \leq m$, *then $J' = A' : I'$ and $\underline{l}$ is regular on R/J. In particular, $R'/(A' : I')$ is Cohen–Macaulay.*

The reader should note that Propositions 1.7(4) and 1.8 hold analogously in the situation where R is a positively graded algebra over a field K with irrelevant maximal ideal M and all the ideals and elements involved are homogeneous. In fact, the passage from R to R_M is a faithfully flat functor on the category of graded modules over R; see Bruns and Herzog [BH, Section 1.5 and especially 1.5.15].

PROPOSITION 1.9. *Let*

$$X = \begin{bmatrix} a_{000}X_0 & \cdots & a_{m-1\,m-1\,0}X_{m-1} \\ \vdots & & \vdots \\ a_{p-1\,0\,p-1}X_{p-1} & \cdots & a_{n-1\,m-1\,p-1}X_{n-1} \end{bmatrix},$$

where the a_{ijk} *are nonzero constants in* K. *Then* $I_p(X)$ *is a perfect ideal of grade* $m - p + 1$ *in* $K[X_0, \dots, X_{n-1}]$.

Proof. By the theorem of Eagon and Northcott (see, e.g., Bruns and Vetter [BV, 2.7]) it is enough to show that ht $I_p(X) \geq m - p + 1$. For $R = K[X_0, \dots, X_{n-1}]$ and $R' = K[X_{p-1}, \dots, X_{m-1}]$ we consider the homomorphism ϕ: $R \to R'$ with $\phi(X_i) = X_i$ for $p - 1 \leq i \leq m - 1$ and $\phi(X_i) = 0$ else. Then $\phi(I_p(X))$ has height $m - p + 1$ by Remark 1.5 (the only difference is that the indeterminates now coincide along the antidiagonals), and therefore $\dim R'/\phi(I_p(X)) = 0$. Furthermore,

$$\begin{aligned} \operatorname{ht} I_p(X) &= \dim R - \dim R/I_p(X) \\ &\geq n - \big(\dim R'/\phi\big(I_p(X)\big) + n - (m - p + 1)\big) = m - p + 1; \end{aligned}$$

note that $\dim R/I_p(X) \leq \dim R'/\phi(I_p(X)) + n - (m - p + 1)$, since the kernel of ϕ is generated by $n - (m - p + 1)$ elements. ∎

PROPOSITION 1.10. *Let* $A = \sum a_{ijk} X_i Y_j Z_k$ *be a nondegenerate diagonal trilinear form of boundary format. Let* X *be as in Proposition* 1.9 *and let* $Z = [Z_0, \dots, Z_{p-1}]$. *Then* $I_1(ZX) + I_p(X)$ *is a geometric* m*-residual intersection of* $(Z_0, \dots, Z_{p-1})$ (*with respect to* $I_1(Z_X)$) *and a perfect ideal.*

Proof. Let

$$U = \begin{bmatrix} U_{00} & \cdots & U_{0m-1} \\ \vdots & & \vdots \\ U_{p-10} & \cdots & U_{p-1\,m-1} \end{bmatrix}$$

be a $p \times m$ generic matrix in the new indeterminates U_{kj}, $0 \leq k \leq p - 1$, $0 \leq j \leq m - 1$. Let $R = K[U, Z_0, \dots, Z_{n-1}]$. By Proposition 1.7 we know that the ideal $J_{p,m} = I_1(ZU) + I_p(U)$ is perfect of grade m. Let L be the ideal generated by the $(p - 1)(m - 1)$ elements $a_{j+k\,j\,k} U_{kj} - a_{j+k\,j-1\,k+1}^{-1} U_{k+1\,j-1}$ with $0 \leq k \leq p - 2$, $1 \leq j \leq m - 1$. If $R' = K[X_0, \dots, X_{n-1}, Z_0, \dots, Z_{p-1}]$, we have $R' \cong R/L$. Now L is a prime ideal with $\operatorname{ht} L = \dim R - \dim R' = mp - n = mp - (m + p - 1) = (p - 1)(m - 1)$, hence its generators form a regular sequence on both R and $R/(Z_0, \dots, Z_{p-1})$. Writing "′" for reduction modulo L we have

$I_1(ZU)' = I_1(ZX)$ and $(Z_0, \dots, Z_{p-1})' = (Z_0, \dots, Z_{p-1})$. If we show that $\mathrm{ht}(I_1(ZX) : (Z_0, \dots, Z_{p-1})) \geq m$ and that $(Z_0, \dots, Z_{p-1})_P = I_1(ZX)_P$ for every prime ideal P containing $(Z_0, \dots, Z_{p-1})$ and of height $\leq m$, we may conclude by the "graded" version of Proposition 1.8 that

$$I_1(ZX) + I_p(X) = \big(I_1(ZX) : (Z_1, \dots, Z_{p-1})\big)$$

and, moreover, that $I_1(ZX) + I_p(X)$ is perfect. Furthermore, $I_1(ZX) + I_p(X)$ is the m-residual intersection of $(Z_0, \dots, Z_{p-1})$ with respect to $I_1(ZX)$.

We start by showing that $\mathrm{ht}(I_1(ZX) : (Z_0, \dots, Z_{p-1})) \geq m$.

Knowing that $I_p(X)(Z_0, \dots, Z_{p-1}) \subseteq I_1(ZX)$ (see the proof of Proposition 1.4), we have $I_1(ZX) + I_p(X) \subseteq (I_1(ZX) : (Z_0, \dots, Z_{p-1}))$. Hence it is enough to show that $\mathrm{ht}(I_1(ZX) + I_p(X)) \geq m$. Let Q be a prime ideal containing $I_1(ZX) + I_p(X)$.

Case 1 ($(Z_0, \dots, Z_{p-1}) \not\subseteq Q$). Let

$$A'_{Y_j} = A_{Y_j} - \big(\text{its monomial summands containing some } Z_k \in Q\big).$$

Hence $A'_{Y_j} \in Q$ for every $j = 0, \dots, m-1$, and A'_{Y_j} is a polynomial in exactly those Z_ks which are outside of Q, say $Z_{k_1}, \dots, Z_{k_t}$ ($k_1 < \cdots < k_t$, $t \geq 1$):

$$A'_{Y_j} = a_{j+k_1 j k_1} X_{j+k_1} Z_{k_1} + \cdots + a_{j+k_t j k_t} X_{j+k_t} Z_{k_t}.$$

Inverting all the indeterminates $Z_{k_1}, \dots, Z_{k_t}$ and denoting by Q^e the extension of the ideal Q, one notices that the elements A'_{Y_j} form a regular sequence in Q^e, and Q^e has the same height was Q. It follows that $\mathrm{ht}\, Q \geq m$.

Actually in this case $\mathrm{ht}\, Q = m$. For if $T = \{Z^t_{k_1}\}_{t \leq 0}$, then $\mathrm{ht}\, Q = \mathrm{ht}\, T^{-1}Q$, and $T^{-1}Q$ is minimal over the ideal $T^{-1}(I_1(ZX) + I_p(X)) = T^{-1}I_1(ZX)$ (we use the fact that $(Z_0, \dots, Z_{p-1})I_p(X) \subseteq I_1(ZX)$). By Krull's principal ideal theorem we conclude that $\mathrm{ht}\, Q = \mathrm{ht}\, T^{-1}Q \leq m$.

It is worth remarking immediately that there always exists a minimal prime of the ideal $I_1(ZX) + I_p(X)$ which does not contain all of the indeterminates $Z_0, \dots, Z_{p-1}$. Otherwise the ideal $(Z_0, \dots, Z_{p-1})$ would be included in the radical of $I_1(ZX) + I_p(X)$, which is in turn included in the ideal $(X_0, \dots, X_{n-1})$.

Case 2 ($(Z_0, \dots, Z_{p-1}) \subseteq Q$). By Proposition 1.9, $I_p(X)$ is a perfect ideal of grade $m - p + 1$. Hence we are done, since then $\mathrm{ht}(Q/(Z_0, \dots, Z_{p-1})) \geq m - p + 1$ and thus $\mathrm{ht}\, Q \geq m + 1$.

In both cases we have obtained that $\operatorname{ht}(I_p(X) + I_1(ZX)) \geq m$. Actually $\operatorname{ht}(I_p(X) + I_1(ZX)) = m$ by the remark at the end of Case 1.

We end the proof of Proposition 1.10 by showing that $(Z_0, \dots, Z_{p-1})_P = I_1(ZX)_P$ for every prime P containing $(Z_0, \dots, Z_{p-1})$ and of height $\leq m$.

Since $I_1(ZX) \subseteq (Z_0, \dots, Z_{p-1})$, it is enough to show that $(Z_0, \dots, Z_{p-1})_P \subseteq I_1(ZX)_P$ for every prime ideal P containing $(Z_0, \dots, Z_{p-1})$ and of height $\leq m$. Clearly $I_p(X) \nsubseteq P$, otherwise P would contain the ideal $(Z_0, \dots, Z_{p-1}) + I_p(X)$ which has height $\geq m + 1$ as proved above. Hence there is a $p \times p$ minor of the matrix X, say f, which is not in P. But $(Z_0, \dots, Z_{p-1})f \subseteq I_1(ZX)$ (recall the proof of Proposition 1.4), and since after localization at P the element f becomes a unit, we get $(Z_0, \dots, Z_{p-1})_P \subseteq I_1(ZX)_P$. ∎

PROPOSITION 1.11. *Let $A = \sum a_{ijk} X_i Y_j Z_k$ be a nondegenerate diagonal trilinear form of boundary format. Then $(Y_0, \dots, Y_{m-1}, A_{Y_0}, \dots, A_{Y_{m-1}}, I_p(X))$ is a perfect prime, hence a minimal prime ideal of J_A.*

Proof. Clearly it is enough to show that, with the notation of Proposition 1.10, $I_1(ZX) + I_p(X)$ is prime; that this ideal is perfect of height m has been proved already. We start by verifying that Z_0 is a regular element modulo $I_p(X) + I_1(ZX)$. We just need to prove that $\operatorname{ht}(I_1(ZX) + I_p(X) + Z_0)) \geq m + 1$ and proceed by induction on $p \geq 2$. Let $p = 2$ and observe that this automatically forces $n = m + 1$. In this case

$$X = \begin{bmatrix} a_{000} X_0 & \cdots & a_{m-1\,m-1\,0} X_{m-1} \\ a_{101} X_1 & \cdots & a_{m\,m-1\,1} X_m \end{bmatrix}$$

and

$$I_1(ZX) + I_p(X) + (Z_0) = \left(a_{101} X_1 Z_1, \dots, a_{m\,m-1\,1} X_m Z_1, I_2(X), Z_0\right).$$

Let Q be any minimal prime over $(a_{101} X_1 Z_1, \dots, a_{m\,m-1\,1} X_m Z_1, I_2(X), Z_0)$. If $Z_1 \notin Q$, then, since all the coefficient are units after localization, $(X_1, \dots, X_m, I_2(X), Z_0) \subseteq Q$ and $\operatorname{ht} Q \geq m + 1$. If $Z_1 \in Q$, then $(Z_0, Z_1, I_2(X)) \subseteq Q$ and, by Proposition 1.9, we again conclude that $\operatorname{ht} Q \geq m + 1$. Let $p > 2$ and assume the statement true for all $p' < p$. Notice that

$$I_1(ZX) + I_p(X) + (Z_0) = I_1(\tilde{Z}\tilde{X}) + I_p(X) + (Z_0),$$

where $\tilde{X}$ stands for the $(p-1) \times m$ matrix

$$\begin{bmatrix} a_{101}X_1 & \cdots & a_{m\,m-1\,1}X_m \\ \vdots & & \vdots \\ a_{p-1\,0\,p-1}X_{p-1} & \cdots & a_{n-1\,m-1\,p-1}X_{n-1} \end{bmatrix}$$

and $\tilde{Z} = [Z_1, \dots, Z_{p-1}]$. Let Q be a prime minimal over $I_1(\tilde{Z}\tilde{X}) + I_p(X) + (Z_0)$. As usual (see the proof of Proposition 1.4) we have $(Z_1, \dots, Z_{p-1})I_{p-1}(\tilde{X}) \subseteq I_1(\tilde{Z}\tilde{X}) \subseteq Q$. If $(Z_1, \dots, Z_{p-1}) \subseteq Q$, then ht $Q \geq m+1$ since $(Z_1, \dots, Z_{p-1}) + I_p(X) \subseteq Q$. If $I_{p-1}(\tilde{X}) \subseteq Q$, then $I_1(\tilde{Z}\tilde{X}) + I_{p-1}(\tilde{X}) + (Z_0) \subseteq Q$ and by induction, ht $Q \geq m+1$.

This concludes the proof that Z_0 is regular modulo $I_1(ZX) + I_p(X)$.

Now, in order to check that $I_1(ZX) + I_p(X)$ is prime, it is enough to notice that $T^{-1}(I_1(ZX) + I_p(X))$ is prime when $T = \{Z_0^t\}_{t \geq 0}$. For $T^{-1}(I_1(ZX) + I_p(X)) = T^{-1}I_1(ZX)$, and $T^{-1}I_1(ZX)$ is obviously a prime ideal. ∎

THEOREM 12. *Let $A = \sum a_{ijk}X_iY_jZ_k$ be a nondegenerate diagonal trilinear form of boundary format. If $p < m$, then the minimal primes of J_A are $(Z_0, \dots, Z_{p-1}, A_{Z_0}, A_{Z_{p-1}})$ and $(Y_0, \dots, Y_{m-1}, A_{Y_0}, \dots, A_{Y_{m-1}}, I_p(X))$.*

Proof. Use Propositions 1.4, 1.5, and 1.11. ∎

Remark 1.13. If $p = m$, it follows by symmetry from Proposition 1.11 that

$$\left(Z_0, \dots, Z_{p-1}, A_{Z_0}, \dots, A_{Z_{p-1}}, I_p(X)\right)$$

is a *minimal* prime ideal of J_A. Here $I_p(X) = (\det(X))$. Since $\det(X)$ does not belong to the ideal $(A_{Z_0}, \dots, A_{Z_{p-1}}) \subseteq (Y_0, \dots, Y_{m-1})$, the ideal $(Z_0, \dots, Z_{p-1}, A_{Z_0}, \dots, A_{Z_{p-1}})$ is properly included in $(Z_0, \dots, Z_{p-1}, I_p(X))$, hence no longer prime. In fact, $Y_j \cdot \det(X) \in (A_{Z_0}, \dots, A_{Z_{p-1}})$ for every $j = 0, \dots, m-1$ (as in the proof of Proposition 1.4).

THEOREM 1.14. *Let $A = \sum a_{ijk}X_iY_jZ_k$ be a nondegenerate diagonal trilinear form of boundary format. If $p = m$, then the minimal primes of J_A are*

$$P_1 = \left(Y_0, \dots, Y_{m-1}, A_{Y_0}, \dots, A_{Y_{m-1}}, I_p(X)\right),$$

$$P_2 = \left(Z_0, \dots, Z_{p-1}, A_{Z_0}, \dots, A_{Z_{p-1}}, I_p(X)\right),$$

and

$$P_3 = \left(Y_0, \dots, Y_{m-1}, Z_0, \dots, Z_{p-1}\right).$$

Proof. It follows by symmetry from Proposition 1.4 that if $p = m$, then a minimal prime Q of J_A, different from P_1 and P_2, must include both $(Z_0, \dots, Z_{p-1}, A_{Z_0}, \dots, A_{Z_{p-1}})$ and $(Y_0, \dots, Y_{m-1}, A_{Y_0}, \dots, A_{Y_{m-1}})$, so it must include $(Y_0, \dots, Y_{m-1}, Z_0, \dots, Z_{p-1})$. But the latter ideal is prime and contains J_A, so that $Q = P_3$. ▮

COROLLARY 1.15. *If* $A = \sum a_{ijk} X_i Y_j Z_k$ *is a nondegenerate diagonal trilinear form of boundary format, then* $\operatorname{ht} J_A = 2p$.

Proof. Since $J_A \subseteq (Z_0, \dots, Z_{p-1}, A_{Z_0}, \dots, A_{Z_{p-1}})$, one has $\operatorname{ht} J_A \leq 2p$, by Krull's principal ideal theorem. By Theorem 1.12 (and the proof of Proposition 1.5) and Theorem 1.14, it is also clear that all the minimal primes of J_A have height greater than or equal to $2p$. ▮

EXAMPLE 1.16. The converse of Corollary 1.15 is false. Let $A = X_0Y_0Z_0 + X_1Y_0Z_1 + X_2Y_1Z_1$. Since $n = 3$, $m = 2 = p$, and $a_{ijk} = 0$ whenever $i \neq j + k$, A is diagonal of boundary format. Using a computer algebra program such as MACAULAY (Bayer and Stillman [BS]), one finds that $\operatorname{ht} J_A = 4 = 2p$. However, several coefficients for which $i = j + k$ are null (e.g., $a_{110} = 0$).

Remark 1.17. (i) Possible generalizations of Theorems 1.12 and 1.14 can take many directions. One can relax any one (or more) of the three assumptions imposed on A: nondegenerate, diagonal, and of boundary format. One can also wonder about t-linear forms for every $t \geq 4$. All generalizations look far from trivial. (ii) The different behavior between case $p = m$ and case $p < m$ has not been detected in the context of hyperdeterminants. It would be nice to understand its meaning from a geometric point of view.

2. THE EMBEDDED ASSOCIATED PRIMES OF J_A

In this section, under the usual assumption, we prove that the maximal ideal of S generated by all the indeterminates is an associated prime of J_A, and we discuss what we expect to be the remaining associated prime ideals.

In order to show that $(X_0, \dots, X_{n-1}, Y_0, \dots, Y_{m-1}, Z_0, \dots, Z_{p-1}) \in \operatorname{Ass} J_A$, we exhibit a ring element $\mu \notin J_A$ such that $(X_0, \dots, X_{n-1}, Y_0, \dots, Y_{m-1}, Z_0, \dots, Z_{p-1})\mu \subseteq J_A$.

LEMMA 2.1. *Let* $A = \sum a_{ijk} X_i Y_j Z_k$ *be a nondegenerate diagonal trilinear form of boundary format. Then* $Y_0^{k+1} Z_k \in J_A$ *for all* k, $0 \leq k \leq p - 1$.

Proof. Let us proceed by induction k. The case $k = 0$ is clear since $a_{000} \neq 0$ and $a_{000} Y_0 Z_0 \in J_A$. Let $k > 0$ and assume $Y_0^{h+1} Z_h \in J_A$ for all h,

$0 \leq h \leq k-1$. Since

$$Y_0^k A_{X_k} = \sum_{l=0}^{k} a_{k\,k-l\,l} Y_0^k Y_{k-1} Z_l$$

is an element of J_A, $Y_0^k Z_l \in J_A$ for all l, $0 \leq l \leq k-1$, by the inductive hypothesis, and $a_{k0k} \leq 0$, we conclude that $Y_0^{k+1} Z_k \in J_A$. ∎

Remark 2.2. The following facts have been noted in Bruns and Guerrieri [BG, Remark 5(c)]: if $A = \sum a_{ijk} X_i Y_j Z_k$ is a nondegenerate diagonal trilinear form of boundary format and J_X denotes the ideal $(A_{X_0}, \ldots, A_{X_{n-1}})$ in the ring $T = K[Y_0, \ldots, Y_{m-1}, Z_0, \ldots, Z_{p-1}]$, then

(1) the elements $\mu = Y_0^{p-1} Z_{p-1}^{m-1}$ and $\mu' = Y_{m-1}^{p-1} Z_0^{m-1}$ are not in J_X;

(2) there exists $a \in K$, $a \neq 0$, such that $\mu \equiv a\mu' \bmod J_X$;

(3) $Y_j \mu$, $Z_k \mu \in J_X$ for all j, k.

These facts are derived from the existence of a monomial K-basis of T/J_X, whose single element in bidegree $(m-1, p-1)$ is μ and that contains no element of bidegree either $(m-1, p)$ or $(m, p-1)$. Moreover, the automorphism of T sending Y_j to Y_{m-1-j} and Z_k to Z_{p-1-k} exchanges μ and μ'.

THEOREM 2.3. *Let $A = \sum a_{ijk} X_i Y_j Z_k$ be a nondegenerate diagonal trilinear form of boundary format. The ideal $(X_0, \ldots, X_{n-1}, Y_0, \ldots, Y_{m-1}, Z_0, \ldots, Z_{p-1})$ is an associated prime of J_A.*

Proof. By Remark 2.2(1), $\mu = Y_0^{p-1} Z_{p-1}^{m-1}$ is an element not in J_A. By considering all the products $Y_0^{p-1} Z_{p-1}^{m-2} A_{Y_j}$ and using Lemma 2.1, we get that $\mu X_i \in J_A$ for all i, $p-1 \leq i \leq n-1$. (Recall that $a_{j+p-1\,j\,p-1} \neq 0$.) By applying the automorphism that sends the variable X_i to the variable X_{n-1-i}, Y_j to Y_{m-1-j}, and Z_k to Z_{p-1-k}, we conclude that $\mu' X_i \in J_A$ for all i, $0 \leq i \leq m-1$. By Remark 2.2(2), we know that $\mu - a\mu' \in J_X$ for some $a \in K$ with $a \neq 0$. It follows that μ is an element not in J_A with $\mu X_i \in J_A$ for all i, $0 \leq i \leq n-1$. Moreover, by Remark 2.2(3), one has $Y_j \mu \in J_X \subseteq J_A$ and $Z_k \mu \in J_X \subseteq J_A$ for all j and k. Thus the maximal homogeneous ideal of S is an associated ideal of J_A. ∎

We conjecture that the embedded associated primes of J_A are precisely the associated primes of the ideals $(Y_0, \ldots, Y_{m-1}, Z_0, \ldots, Z_{p-1}, I_t(X))$, with $1 \leq t \leq p-1$, where X is the matrix described in Proposition 1.4. This conjecture has been suggested by an ample experimental evidence based on calculations with MACAULAY [BS]. Furthermore, it is not hard to see that every associated prime ideal different from the minimal ones must contain $(Y_0, \dots, Y_{m-1}, Z_0, \dots, Z_{p-1})$. If all a_{ijk} are 1, X is a Hankel matrix and the ideals $I_t(X)$ are prime (see, for example, Eisenbud [E]). However, $I_t(X)$ is not a prime ideal in general, so that the coefficients a_{ijk} cannot be neglected in the determination of the remaining associated prime ideals of J_A.

REFERENCES

[A] L. L. Avramov, Complete intersections and symmetric algebras, *J. Algebra* **73** (1981), 248–263.

[BS] D. Bayer and M. Stillman, Macaulay: A system for computation in algebraic geometry and computation algebra, available via anonymous `ftp` from `math.harvard.edu`.

[BG] W. Bruns and A. Guerrieri, The Dedekind–Mertens formula na determinantal rings, *Proc. Amer. Math. Soc.*, to appear.

[BH] W. Bruns and J. Herzog, "Cohen–Macaulay Rings," Cambridge Univ. Press, Cambridge, UK, 1993.

[BKM] W. Bruns, A. Kustin, and M. Miller, The resolution of the generic residual intersection of a complete intersection, *J. Algebra* **128** (1990), 214–239.

[BV] W. Bruns and U. Vetter, "Determinantal rings," Springer Lecture Notes in Mathematics, Vol. 1327, Springer-Verlag, Berlin/New York, 1988.

[E] D. Eisenbud, Linear sections of determinantal varieties, *Amer. J. Math.* **110** (1988), 541–575.

[GKZ] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky, Hyperdeterminants, *Adv. in Math.* **96** (1992), 226–263.

[H] C. Huneke, On the symmetric algebra of a module, *J. Algebra* **69** (1981), 113–119.

[HU] C. Huneke and B. Ulrich, Residual intersections, *J. Reine Angew. Math.* **390** (1988), 1–20.

[SV] A. Simis and W. V. Vasconcelos, On the dimension and integrality of symmetric algebras, *Math. Z.* **177** (1981), 341–358.

[WZ] J. Weyman and A. V. Zelevinsky, Singularities of hyperdeterminants, preprint (1994).